

Information and Distinguishability of Ensembles of Identical Quantum States

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We consider a fixed quantum measurement performed over n identical copies of quantum states. Using a rigorous notion of distinguishability based on Shannon's 12th theorem, we show that in the case of a single qubit, the number of distinguishable states is $W(\alpha_1, \alpha_2, n) = |\alpha_1 - \alpha_2| \sqrt{\frac{2n}{\pi e}}$, where (α_1, α_2) is the angle interval from which the states are chosen. In the general case of an N -dimensional Hilbert space and an area Ω of the domain on the unit sphere from which the states are chosen, the number of distinguishable states is $W(N, n, \Omega) = \Omega \left(\frac{2n}{\pi e}\right)^{\frac{N-1}{2}}$. The optimal distribution is uniform over the domain in Cartesian coordinates.

KEY WORDS: information in quantum measurements; distinguishability of quantum states; number of distinguishable quantum states; information criterion for distinguishability.

1. INTRODUCTION

In his remarkable 1981 paper, "Statistical Distance and Hilbert Space" (Wootters, 1981), Wootters showed that the statistical distance between two vectors in Hilbert space is proportional to the angle between these two vectors and does not depend on the position of the vectors. He defines statistical distance as the number of distinguishable intermediate states between the two vectors. However, his notion of distinguishability relies on the apparently arbitrary criterion that two states are distinguishable if measurements performed on n identical copies of each state yield two distributions whose means are separated by a constant factor times the sum of the standard deviations of these distributions. We use a more rigorous notion of distinguishability based on Shannon's 12th theorem (Shannon and Weaver, 1949) and arrive at an expression for the number of distinguishable states that is consistent with Wootters' result; however, unlike that result, our expression does not depend on an arbitrary choice of the distinguishability criterion. Rather,

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our notion of distinguishability is predicated on the guarantee that the measurer be able to distinguish between the quantum states with probability approaching 1 when n copies of identical states are used as a symbol in a signal sequence whose length tends to infinity. Wootters shows that for large n the number of distinguishable states between the vectors α_1 and α_2 is proportional to $|\alpha_1 - \alpha_2|\sqrt{n}$, where α is the angle of the vector from some reference direction in the plane spanned by the two vectors. We show in Section 2 that the actual number of distinguishable states in a two-dimensional Hilbert space is

$$W(\alpha_1, \alpha_2, n) = e^{I_{\text{sup}}(P;K)} = |\alpha_1 - \alpha_2| \sqrt{\frac{2n}{\pi e}}, \quad (1)$$

where $I_{\text{sup}}(P; K)$ is the maximum mutual information between the (random) quantum state and the results of measurements. We prove that this maximum is achieved for an ensemble of quantum states with the uniform distribution of the angle α for any interval $[\alpha_2, \alpha_1]$. The independence of the number of distinguishable states of the position of the interval $[\alpha_2, \alpha_1]$ is a remarkable asymptotic property that does not hold for small values of n (cf. Levitin, 1994).

Section 3 of this paper provides a generalization of these results to the case of an N -dimensional Hilbert space of states of the quantum system. It turns out that the number of distinguishable states depends only on the area Ω of the domain on the unit sphere from which the states can be chosen, but does not depend on the shape and position of this domain. The optimal distribution is uniform over this domain in Cartesian coordinates, and the number of distinguishable states is $W(N, n, \Omega) = \Omega \left(\frac{2n}{\pi e}\right)^{(N-1)/2}$.

2. THE CASE OF A SINGLE QUBIT

2.1. Formulation of the Problem

Consider a quantum physical system whose states are unit vectors in a two-dimensional complex Hilbert space \mathbf{C}^2 (the so-called ‘‘qubit’’). Denote the state vector by \mathbf{v} and let (ϕ, ψ) be an orthonormal basis in the Hilbert space, so that $\mathbf{v} = a\phi + b\psi$, where $a = \langle \mathbf{v} | \phi \rangle$, $b = \langle \mathbf{v} | \psi \rangle$ are inner products and $|a|^2 + |b|^2 = 1$. Then $|a|^2 = p$ and $|b|^2 = 1 - p$ are probabilities of two possible outcomes of the measurement performed over the state \mathbf{v} in the (ϕ, ψ) basis. Obviously, these probabilities do not depend on the phases of the coefficients a and b , and, therefore, all quantum states with the same magnitudes $|a| = x$ and $|b| = y$ are indistinguishable by this measurement. Hence, the state space \mathbf{S}^3 can be reduced to the non-negative quadrant of a circle in a real 2-dimensional Euclidean space (Fig. 1), spanned by ϕ and ψ . Now let \mathbf{v}_1 and \mathbf{v}_2 be two distinct state vectors, such that

$$\mathbf{v}_i = x_i\phi + y_i\psi \quad \text{where} \quad x_i = \sqrt{p_i} \quad \text{and} \quad y_i = \sqrt{1 - p_i}, \quad \text{for } i = 1, 2. \quad (2)$$

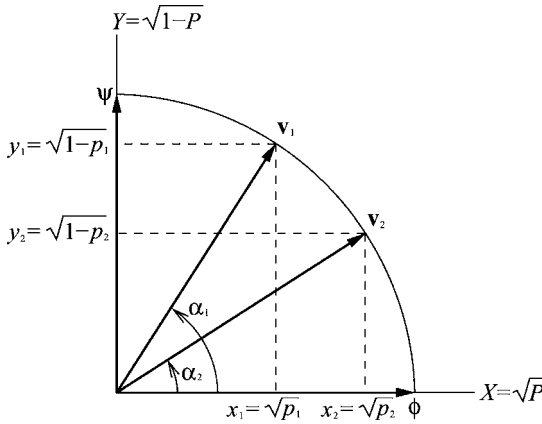


Fig. 1. The two state vectors v_1, v_2 and their projections on the the basis elements ϕ and ψ .

Denote by α_i the angle between ϕ and v_i , so that

$$p_i = \cos^2 \alpha_i, \quad 1 - p_i = \sin^2 \alpha_i, \quad i = 1, 2. \tag{3}$$

Suppose, we want to distinguish between various quantum states chosen from the interval of angles $[\alpha_2, \alpha_1]$ by performing measurements in the (ϕ, ψ) basis. Further, assume that we are allowed to perform the measurement over n identical copies of each quantum state.

Problem: What determines the number of distinguishable states, and what is the asymptotic expression for the number of states in the interval $[\alpha_2, \alpha_1]$ that can be distinguished with probability approaching 1 when n tends to infinity?

As shown in the next section, the problem can be rigorously analyzed by applying concepts and results of Shannon’s information theory.

2.2. Information-Theoretical Description

Suppose the state vectors are chosen from the angle interval $[\alpha_2, \alpha_1]$ with certain probability density function (pdf) $P_A(\alpha)$, where A is a random variable that takes on values from $[\alpha_2, \alpha_1], \alpha \in [\alpha_2, \alpha_1]$. Let $P_P(p)$ be the pdf of the random variable P that takes on values p , where p is the probability of the state vector to be projected as the result of the measurement onto basis vector ϕ . Obviously, $P = \cos^2 A$, and the value of P (or of A) characterizes uniquely the chosen quantum state. In a series of n measurements, let K be the (random) number of measurements which have resulted in projections onto ϕ . The conditional probability distribution of K given P is binomial:

$$P_{K/P}(k/p) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad \text{where } k = 0, 1, \dots, n. \tag{4}$$

The values of K obtained in the measurement are the only data available from which one can infer about the value of P , i.e., about the choice of a quantum state.

Let $P_K(k)$ be the marginal probability distribution of K . The information $I(K; P)$ in K about P is given by

$$I(K; P) = \int_{p_1}^{p_2} \sum_{k=0}^n P_P(p) P_{K/P}(k/p) \ln \frac{P_{K/P}(k/p)}{P_K(k)} dp. \quad (5)$$

The importance of considering information $I(K, P)$ stems from Shannon's 12th theorem (Shannon and Weaver, 1949) which, for our setting of the problem, can be rephrased in the following way.

Let $S = \{\mathbf{p}\}$, where \mathbf{p} is an n -dimensional vector $\mathbf{p} = (p, p, \dots, p)$ and $p \in [p_1, p_2]$ be the set of all possible input signals and $\mathbf{Z}_{n+1} = \{0, 1, \dots, n\}$ be the set of all output signals in a communication channel with a conditional probability distribution given by (4). Let L be the length of a sequence of such input signals used independently. Then for any $\varepsilon > 0$ the maximum number $M(L, \varepsilon)$ of input signals that can be chosen from S in such a way that the probability of error (incorrect decision about \mathbf{p} based on the value of the output signal $k \in \mathbf{Z}_{n+1}$) does not exceed ε satisfies the asymptotic property:

$$\lim_{L \rightarrow \infty} \left[\frac{\ln M(L, \varepsilon)}{L} \right] = I_{\text{sup}}(K; P), \quad (6)$$

where $I_{\text{sup}}(K; P)$ is the least upper bound of $I(K; P)$ given by (5) over all possible probability distributions $P_P(p)$ of the input parameter P .

Note that the asymptotic expression for $M(L, \varepsilon)$ in fact does not depend on ε . This means that the number of distinct input signals (different values of P) that can be distinguished with probability arbitrarily close to 1 is $e^{I_{\text{sup}}(K; P)}$. The problem is reduced now to the computation of $I_{\text{sup}}(K; P)$ under the condition that P takes on values in $[p_1, p_2]$. This problem is very difficult, in general. However, the following important theorem will be helpful.

Define individual information in $P = p$ about K as

$$I(K; p) = \sum_{k=0}^n P_{K/P}(k/p) \ln \frac{P_{K/P}(k/p)}{P_K(k)}. \quad (7)$$

As is well known (e.g. Mansuripur, 1987), $I(K; P)$ achieves the maximum value $I_{\text{sup}}(K; P)$ for such a distribution $P_P(p)$ that there exists a constant I such that

$$I(K; p) = I \quad \text{for all } p \quad \text{such that } P_P(p) > 0 \quad (8)$$

and

$$I(K; p) < I \quad \text{for all } p \quad \text{such that } P_P(p) = 0. \quad (9)$$

Then $I_{\text{sup}}(K; P) = I$.

2.3. The Number of Distinguishable States

When n is large, the binomial distribution (4) can be well approximated by a Gaussian distribution:

$$P_{K/P}(k/p) = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{1}{\sqrt{2\pi p(1-p)n}} e^{-\frac{(k-pn)^2}{2p(1-p)n}}. \quad (10)$$

For large n , distribution (10) has a very sharp maximum at $k = pn$, so that the Laplace method (De Bruijn, 1958) can be used for evaluation of integrals involving (10).

Consider a uniform distribution over the angle interval $[\alpha_2, \alpha_1]$,

$$P_A(\alpha) = \frac{1}{|\alpha_1 - \alpha_2|} \quad \text{for } \alpha \in [\alpha_2, \alpha_1]. \quad (11)$$

The corresponding distribution of the probability P is

$$P_P(p) = P_A(\alpha) \left| \frac{d\alpha}{dp} \right| = \frac{[p(1-p)]^{-1/2}}{2|\alpha_1 - \alpha_2|} \quad \text{where } \alpha_i = \cos^{-1} \sqrt{p_i}, \quad i = 1, 2. \quad (12)$$

We will prove that for large n this distribution yields the maximum of $I(K; P)$. The marginal probability distribution $P_K(k)$ can be evaluated as follows (assuming $p_2 > p_1$):

$$\begin{aligned} P_K(k) &= \int_{p_1}^{p_2} P_P(p) P_{K/P}(k/p) dp \\ &\approx \frac{1}{2|\alpha_1 - \alpha_2|} \int_{p_1}^{p_2} \frac{1}{p(1-p)\sqrt{2\pi n}} e^{-\frac{(k-pn)^2}{2p(1-p)n}} dp. \end{aligned} \quad (13)$$

If the point of maximum $p = \frac{k}{n}$ of the exponential function in the integrand is within the interval $[p_1, p_2]$, the integration interval can be extended to $(-\infty, \infty)$. Otherwise, the value of the integral approaches zero when n tends to infinity. Thus, for large n we obtain:

$$P_K(k) \approx \begin{cases} \frac{1}{2|\alpha_1 - \alpha_2| \sqrt{k(n-k)}} & \text{if } np_1 \leq k \leq np_2, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Note that, as could be expected, the distribution of K for large n is the discrete counterpart of the distribution of P . Now we can evaluate the individual

information $I(K; p)$:

$$\begin{aligned} I(K; p) &\approx \sum_{k=\lceil np_1 \rceil}^{\lfloor np_2 \rfloor} P_{K/P}(k/p) \ln \frac{P_{K/P}(k/p)}{P_K(k)} \\ &\approx \int_{np_1}^{np_2} \frac{dk}{p(1-p)\sqrt{2\pi n}} e^{-\frac{(k-np)^2}{2p(1-p)n}} [\ln P_{K/P}(k/p) - \ln P_K(k)]. \end{aligned} \quad (15)$$

The first term in (15) is the differential entropy of a Gaussian distribution (with the opposite sign), the second one can be evaluated by the Laplace method. Hence, asymptotically,

$$\begin{aligned} I(K; p) &= -\frac{1}{2} \ln[2\pi e p(1-p)n] + \frac{1}{2} \ln[p(1-p)n^2] + \ln 2|\alpha_1 - \alpha_2| \\ &= \frac{1}{2} \ln \frac{2n}{\pi e} + \ln |\alpha_1 - \alpha_2|. \end{aligned} \quad (16)$$

Note that $I(K; p)$ takes on the same value for any $p \in [p_1, p_2]$. Hence, distribution (11) (or (12)) is the optimal one for large n , and the maximum information $I_{\text{sup}}(K; P)$ is expressed asymptotically as

$$I_{\text{sup}}(K; P) = \frac{1}{2} \ln \frac{2n}{\pi e} + \ln |\alpha_1 - \alpha_2|. \quad (17)$$

Thus, the number of distinguishable quantum states in the interval of angles $[\alpha_2, \alpha_1]$ is proportional to the length of the interval and to \sqrt{n} . It does not depend on the position of the interval in the circle.

$$W(n, \alpha_1, \alpha_2) = e^{I_{\text{sup}}(K; P)} = |\alpha_1 - \alpha_2| \sqrt{\frac{2n}{\pi e}}. \quad (18)$$

Of course, the range of A may consist of several separated intervals. Then (18) remains valid, as long as n is sufficiently large, so that each interval has many distinguishable states; also, $|\alpha_1 - \alpha_2|$ should be replaced by the total length of the intervals.

For given n , (18) achieves maximum if $|\alpha_1 - \alpha_2| = \pi/2$. Hence,

$$W_{\text{max}}(n) = \sqrt{\frac{\pi n}{2e}}. \quad (19)$$

3. THE N -DIMENSIONAL CASE

Consider now a quantum system whose states are unit vectors in an N -dimensional complex Hilbert space \mathbf{C}^N . Choose an orthogonal basis in \mathbf{C}^N corresponding to a direct (von Neumann) measurement. Since all quantum states having

the same projections on the basis vectors are indistinguishable by this measurement, the state space \mathbf{S}^{2N-1} is reduced to the non-negative orthant of the unit sphere \mathbf{S}^{N-1} in the real N -dimensional Euclidean space \mathbf{R}^N . Each state vector is described now by N Cartesian coordinates $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $\sum_{i=1}^N x_i^2 = 1$, and $p_i = x_i^2$ is the probability of the i th outcome of the measurement. Suppose we want to distinguish between states chosen from a domain \mathbf{D} of the non-negative orthant of \mathbf{S}^{N-1} , and assume we are allowed to perform the same measurement over n identical copies of each quantum state, where $n \gg 1$. Let the quantum states be chosen with probability density function $\mathbf{P}_{\mathbf{P}}(\mathbf{p}) = \mathbf{P}_{\mathbf{P}}(p_1, \dots, p_N)$, where $\sum_{i=1}^N p_i = 1$. The outcome of such a measurement performed over n identical states is an N -dimensional random variable \mathbf{K} which takes on values $\mathbf{k} = (k_1, k_2, \dots, k_N)$, where k_i ($i = 1, 2, \dots, N$) is the number of cases when the i th result has been obtained. The conditional probability distribution of \mathbf{K} given \mathbf{P} is multinomial:

$$\mathbf{P}_{\mathbf{K}/\mathbf{P}}(k_1, \dots, k_m / p_1, \dots, p_m) = \frac{n!}{\prod_{i=1}^N k_i!} \prod_{i=1}^N p_i^{k_i}, \tag{20}$$

where $\sum_{i=1}^N k_i = n$.

Denote by $\mathbf{P}_{\mathbf{K}}(\mathbf{k})$ the marginal probability distribution of \mathbf{K} . Then the information $\mathbf{I}(\mathbf{K}; \mathbf{P})$ in \mathbf{K} about \mathbf{P} is given by an expression similar to (5):

$$\mathbf{I}(\mathbf{K}; \mathbf{P}) = \int_{\mathbf{p} \in \mathbf{D}} \sum_{\mathbf{k}} \mathbf{P}_{\mathbf{P}}(\mathbf{p}) \mathbf{P}_{\mathbf{K}/\mathbf{P}}(\mathbf{k}/\mathbf{p}) \ln \frac{\mathbf{P}_{\mathbf{K}/\mathbf{P}}(\mathbf{k}/\mathbf{p})}{\mathbf{P}_{\mathbf{K}}(\mathbf{k})} d\mathbf{p}, \tag{21}$$

where summation is taken over all \mathbf{k} such that $\sum_{i=1}^N k_i = n$.

It follows from Shannon's 12th theorem that for any $\varepsilon > 0$ the maximum number of distinct states $\mathbf{W}(N, n, \varepsilon)$ chosen from \mathbf{D} in such a way that the probability of incorrect identification of the state based on the results \mathbf{K} of the measurement does not exceed ε satisfies the limit

$$\lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\ln \mathbf{W}(N, n, \varepsilon)}{\mathbf{I}_{\text{sup}}(\mathbf{K}; \mathbf{P})} = 1. \tag{22}$$

Here $\mathbf{I}_{\text{sup}}(\mathbf{K}; \mathbf{P})$ is the least upper bound of $\mathbf{I}(\mathbf{K}; \mathbf{P})$ over all possible $\mathbf{P}_{\mathbf{P}}(\mathbf{p})$. Note that, in contrast with the two-dimensional case, there is no need to consider sequences of distinct states provided n and N are sufficiently large.

Thus the number of distinct states (different values of \mathbf{P}) that can be distinguished with probability arbitrarily close to 1 is given by $e^{\mathbf{I}_{\text{sup}}(\mathbf{K}; \mathbf{P})}$. The computation of $\mathbf{I}_{\text{sup}}(\mathbf{K}; \mathbf{P})$ can be performed along the same lines as in the two-dimensional case. For large n ($n/N \gg 1$), the multinomial distribution (20) can be approximated by the N -dimensional Gaussian distribution (Gnedenko, 1962)

$$\mathbf{P}_{\mathbf{K}/\mathbf{P}}(\mathbf{k}/\mathbf{p}) \approx \frac{e^{-\frac{1}{2} \sum_{i=1}^N \frac{(k_i - p_i n)^2}{p_i n}} \delta(\sum_{i=1}^N k_i - n)}{(2\pi n)^{\frac{N-1}{2}} \prod_{i=1}^N p_i^{\frac{1}{2}}}. \tag{23}$$

Consider the distribution $P_{\mathbf{X}}(\mathbf{x})$ of the states which is uniform over the domain \mathbf{D} . Denote the area of \mathbf{D} by $|\mathbf{D}| = \Omega$. Then

$$P_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\Omega} \delta \left(\sqrt{\sum_{i=1}^N x_i^2} - 1 \right) \quad (24)$$

for $\mathbf{x} \in \mathbf{D}$, and $p_{\mathbf{X}}(\mathbf{x}) = 0$ otherwise. We will show that for large n this distribution yields the maximum of $I(\mathbf{K}; \mathbf{P})$. Distribution (24) corresponds to the following distribution of the random variable \mathbf{P} over the domain \mathbf{D} :

$$P_{\mathbf{P}}(p_1, \dots, p_N) = \frac{1}{\Omega} J \left(\frac{x_1, \dots, x_N}{p_1, \dots, p_N} \right) \delta \left(\sqrt{\sum_{i=1}^N p_i} - 1 \right) \quad (25)$$

$$= \frac{\prod_{i=1}^N p_i^{-\frac{1}{2}} \delta \left(\sum_{i=1}^N p_i - 1 \right)}{2^{N-1} \Omega}, \quad (26)$$

where $J \left(\frac{x_1, \dots, x_N}{p_1, \dots, p_N} \right)$ is the Jacobian of the coordinate transformation from \mathbf{x} to \mathbf{p} . The marginal probability distribution of \mathbf{K} is given by

$$P_{\mathbf{K}}(k_1, \dots, k_N) = \int_{\mathbf{D}} P_{\mathbf{P}}(\mathbf{p}) P_{\mathbf{K}/\mathbf{P}}(\mathbf{k}/\mathbf{p}) dp_1 \dots dp_N. \quad (27)$$

For large n , the integrand in (27) has a sharp maximum at $\mathbf{p} = \mathbf{k}/n$. Applying again the Laplace method we obtain:

$$P_{\mathbf{K}}(k_1, \dots, k_N) \approx \frac{\prod_{i=1}^N \left(\frac{k_i}{n} \right)^{-\frac{1}{2}} \delta \left(\sum_{i=1}^N k_i - n \right)}{2^{N-1} \Omega}, \quad (28)$$

when $\frac{\mathbf{k}}{n}$ corresponds to a point in the domain \mathbf{D} ; otherwise $P_{\mathbf{K}}(\mathbf{k}) = 0$. The individual information $I(\mathbf{K}; \mathbf{p})$ can be conveniently evaluated by use of “reduced” distributions $P_{\mathbf{K}'/\mathbf{P}}(\mathbf{k}'/\mathbf{p})$ and $p_{\mathbf{K}'}(\mathbf{k}')$, where we take into account explicitly the dependence between the components of the vector \mathbf{k} implied by the δ -function:

$$P_{\mathbf{K}'/\mathbf{P}} = \frac{\exp \left[-\frac{1}{2} \sum_{i=1}^{N-1} \frac{(k_i - p_i n)^2}{p_i n} - \frac{(1 - p_m n - \sum_{i=1}^{N-1} k_i)^2}{2 p_m n} \right]}{(2\pi n)^{\frac{N-1}{2}} \prod_{i=1}^N p_i^{\frac{1}{2}}}, \quad (29)$$

$$P_{\mathbf{K}'}(\mathbf{k}') = \frac{\left(n - \sum_{i=1}^{N-1} k_i \right)^{-\frac{1}{2}} \prod_{i=1}^{N-1} k_i^{-\frac{1}{2}}}{2^{(N-1)} \Omega n^{\frac{N}{2}-1}}. \quad (30)$$

Then

$$\begin{aligned}
 I(\mathbf{K}; \mathbf{p}) = I(\mathbf{K}'; \mathbf{p}) &= \int_{\frac{\mathbf{k}}{n} \in \mathbf{D}} P_{\mathbf{K}'/\mathbf{P}}(\mathbf{k}'/\mathbf{p}) [\ln P_{\mathbf{K}'/\mathbf{P}}(\mathbf{k}'/\mathbf{p}) \\
 &\quad - \ln P_{\mathbf{K}'}(\mathbf{k}')] dk_1 \dots dk_{N-1} = I_1 + I_2. \tag{31}
 \end{aligned}$$

The first term in (31) is simply the differential entropy (with the opposite sign) of a multivariate $(N - 1)$ -dimensional Gaussian distribution (29) with the determinant of covariance matrix $d = n^{N-1} \prod_{i=1}^N p_i$. Hence

$$I_1 = -\frac{1}{2} \ln [(2\pi e)^{N-1} d] = -\frac{1}{2} \ln \left[(2\pi en)^{N-1} \prod_{i=1}^N p_i \right]. \tag{32}$$

The second term in (31) can be evaluated by the Laplace method, since the integrand has a sharp maximum at $k_i = p_i n$ ($i = 1, \dots, N - 1$). Hence

$$I_2 = \ln \Omega + \ln(2n)^{N-1} + \frac{1}{2} \ln \prod_{i=1}^N p_i. \tag{33}$$

Thus, $I(\mathbf{K}; \mathbf{p}) = \ln \Omega + \frac{N-1}{2} \ln \frac{2n}{\pi e}$ does not depend on \mathbf{p} . This proves that the distribution (24) is the optimal one and the maximum information in \mathbf{K} about \mathbf{P} is asymptotically equal to

$$I_{\text{sup}}(\mathbf{K}; \mathbf{P}) = \ln \Omega + \frac{N-1}{2} \ln \frac{2n}{\pi e}. \tag{34}$$

The number of distinguishable states is given by the following expression:

$$W(N, n, \Omega) = \Omega \left(\frac{2n}{\pi e} \right)^{\frac{N-1}{2}}. \tag{35}$$

Expression (35) turns into (18) for $N = 2$. Indeed, it is easy to see that a uniform two-dimensional distribution in Cartesian coordinates restricted to the non-negative quadrant of a unit circumference results in a uniform distribution over the polar angle α . Similarly, in the N -dimensional case we obtain a uniform distribution over the area of the domain \mathbf{D} , i.e. over the solid angle.

The number of distinguishable states reaches a maximum (for given N and n) if \mathbf{D} is the entire non-negative orthant of the N -dimensional unit sphere. Since the area of the surface of the N -dimensional unit sphere is $2\pi^{N/2} [\Gamma(N/2)]^{-1}$, the area of the non-negative orthant (the solid angle) is

$$\Omega_{\text{max}} = \frac{\pi^{N/2}}{2^{N-1} \Gamma(N/2)}, \tag{36}$$

where Γ is Euler's gamma-function. Thus the maximum number of distinguishable states in N -dimensional space is

$$W_{\max}(N, n) = \frac{\pi^{1/2}}{\Gamma(N/2)} \left(\frac{n}{2e} \right)^{\frac{N-1}{2}}. \quad (37)$$

Remember that (36) and (37) are valid only when approximation (23) is valid, i.e. when $n \gg N$.

4. CONCLUSIONS

The main result of the paper can be summarized as follows. The number of distinguishable quantum states in a two-dimensional Hilbert space is proportional to the number of identical copies of each state to the power $\frac{N-1}{2}$ and to the area Ω of the domain of the unit sphere occupied by the state vectors. Surprisingly, it does not depend on the shape and the position of this domain, provided that the main assumption $n/N \gg 1$ is satisfied. The domain does not have to be connected: the results hold for a set of separate domains with the same total area Ω . The optimal distribution is uniform over the domain, which suggests that the states should be chosen at equal angular distances from each other. For the two-dimensional case, the number of distinguishable states is proportional to the angular interval and to the square root of the number of identical copies of each state measured (cf. Levitin *et al.*, 2001).

The result that the number of distinguishable states is proportional to the geometric distance as measured by angle in Hilbert space is quite nontrivial and noteworthy. Indeed, it suggests that the metric of Hilbert space may result not from a physical principle, but rather as a consequence of an optimal statistical inference procedure.

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REFERENCES

- De Bruijn, N. G. (1958). *Asymptotic Methods in Analysis*, North Holland Publishing Co., Chapter 4.
- Gnedenko, B. V. (1962). *The Theory of Probability*, Chelsea, New York, p. 85.
- Levitin, L. B. (1994). *Open Systems and Information Dynamics*, Vol. 2, pp. 319–329.
- Levitin, L. B., Toffoli, T., and Walton, Z. (2001). In *Quantum Communication, Computing, and Measurement*, P. Tombesi and O. Hirota, eds., Kluwer Academic, New York, Vol. 3, pp. 19–25.
- Mansuripur, M. (1987). *Introduction to Information Theory*, Prentice-Hall, p. 66.
- Shannon, C. E. and Weaver, W. (1949). *The Mathematical Theory of Communication*, University of Illinois Press, p. 76.
- Wooters, W. K. (1981). *Phys. Rev. D*, **23**, 357–362.